THE WIENER INTEGRAL AND THE SCHRÖDINGER OPERATOR(1)

BY
DONALD BABBITT

1. **Introduction.** In non-relativistic quantum mechanics the Schrödinger operator is of paramount importance. It is of the form

$$A(b;\, V;\mathfrak{D})\phi = \sum_{j=1}^{n} \frac{1}{2}(i\partial_j + b_j(x))^2\phi + V(x)\phi, \tag{1.1}$$

where $x = (x_1, \cdots, x_n) \in R^n, b_j(x),\ j = 1, \cdots, n$ and $V(x)$ are real-valued functions on R^n, $\partial_j = \partial/(\partial x_j)$, $i = \sqrt{-1}$ and $\phi \in \mathfrak{D}$, a dense linear subset of $L_2(R^n)$. $\mathfrak{D}$ is called the domain of $A(b;\, V;\mathfrak{D})$ and is denoted by $\mathfrak{D}(A)$. The domain $\mathfrak{D}$ and/or $(b;\, V)$ will be omitted from $A(b;\, V;\mathfrak{D})$ when no confusion should arise. In practice, the spectral resolution of a certain self-adjoint extension, $\tilde{A}$, of A is important.

One important technique used in studying pertinent properties of the spectral resolution of $\tilde{A}(b;\, V;\mathfrak{D})$ is to represent $e^{-\tilde{A}t}\phi,\ \phi \in L_2(R^n)$, by the Wiener integral of a certain functional $F(b;\, V;\phi)$ and then to translate the (at least formally) transparent properties of this integral into less transparent but important properties of $\tilde{A}$. For example, see [14] and [18]. This functional integral representation has been shown to be rigorously valid for the case where $b_j(x) = 0\ (j = 1, \cdots, n)$, and where $V(x)$ is Borel measurable and bounded from below. Unfortunately, $\tilde{A}$ is only given implicitly. See [8], [9], and [10]. It is the purpose of this paper to extend the validity of the functional integral representation of $e^{-\tilde{A}t}\phi$ to the case where $b(x) = (b_1(x), \cdots, b_n(x))$ is a sufficiently smooth vector field which doesn't grow too rapidly at infinity, and where $V(x)$ can have certain negative singularities. Moreover, $\tilde{A}$ will be given explicitly(2). The Schrödinger operator containing (normal) Zeeman effect and Coulomb interaction terms will be included.

§§2, 3, and 4 will give a brief account of functional integration, semigroup theory and quadratic forms on a Hilbert space, respectively. They will be needed for the main body of the paper consisting of §§5 and 6. §5 will give the statement of the theorems along with certain remarks. §6 will contain the proofs of the theorems.

Received by the editors April 14, 1964.

(1) The preparation of this paper was sponsored (in part) by the National Science Foundation grant GP-1988.

(2) The notion of local operators discussed in [8] does not apply to the situation at hand.

We, will not, in general, repeat an argument already contained in another source but will refer the reader to that source. In particular, results from [3], [4], [8], [9], and [14] will be referred to extensively.

We would like to thank Professor E. Nelson for suggesting many improvements to Theorem 1 in [1]. These suggestions led directly to the writing of this paper.

2. **Functional integration.** Let $W = W(R^n)$ be the space of continuous functions from $[0, \infty)$ to R^n. Let $\mathfrak{N}^t$ be the σ-field of subsets of W generated by sets of the form $\{x(\cdot) \in W \mid x(s) \in \Gamma\}$, where $0 \leq s \leq t$ and Γ is a Borel measurable subset of R^n. Let $\mathfrak{N} = \bigcup_{t \geq 0} \mathfrak{N}^t$. Then $(W, \mathfrak{N})$ is a measurable space which is often called Wiener space.

Let $p^\sigma(t,x,y)$ be the fundamental solution (assuming $p^\sigma(t,x,y)$ satisfies the usual boundary conditions [19]) of the following pair of parabolic equations:

$$\frac{\partial p^\sigma}{\partial t} = \frac{1}{2}\Delta_x p^\sigma - \sigma(b(x) \cdot \nabla_x)p^\sigma, \quad \frac{\partial p^\sigma}{\partial t} = \frac{1}{2}\Delta_y p^\sigma + \sigma \nabla_y \cdot (b(x)p^\sigma),$$

where Δ is the Laplacian $\sum_{j=1}^n (\partial_j)^2$, ∇ is the gradient operator $(\partial_1, \cdots, \partial_n)$, $b(\cdot)$ is a bounded twice continuously differentiable vector field on R^n and σ is a real parameter (note that $b \cdot \nabla \equiv \sum_{j=1}^n b_j(x)\partial_j$ and $\nabla \cdot (bp) = \sum_{j=1}^n \partial_j(b_j p)$). Then there exists a family of measures $\{P_x^\sigma \mid x \in R^n\}$ on $(W, \mathfrak{N})$ such that, for any finite subdivision $0 = t_0 < t_1 < \cdots < t_r$ of $[0, \infty)$ and any sequence $\Gamma_1, \cdots, \Gamma_r$ of Borel subsets of R^n $(r = 1, 2, \cdots)$, we have

$$\int_{\Gamma_1 \times \cdots \times \Gamma_r} \prod_{j=1}^r p^\sigma(t_j - t_{j-1}, x_j, x_{j-1})\, dx_1 \cdots dx_r = P_x^\sigma\{x(\cdot) \in W \mid x(t_j) \in \Gamma_j; j = 1, \cdots, r\},$$

where $x_0 \equiv x$. The existence of this family of measures follows from the fact that $p^\sigma(t,x,y)$ determines a stationary Markov process with continuous paths(3). See [6] for further references.

To assure the measurability of certain pertinent functionals we "complete" the $\mathfrak{N}^t$ with respect to the $\{P_x^0 \mid x \in R^n\}$. We denote this "completion" by $\bar{\mathfrak{N}}^t$ and let $\bar{\mathfrak{N}} = \bigcup_{t \geq 0} \mathfrak{N}^t$(4). See [4] for a more detailed discussion of this notion of completion.

The following functionals on W are $\bar{\mathfrak{N}}$-measurable and will be of interest in the sequel.

A. $F^1_{\phi t}(x(\cdot)) \equiv \phi(x(t))$, where $\phi \in L_2(R^n)$ and $t > 0$.

B. $F^2_{Vt}(x(\cdot)) \equiv \exp[-\int_0^t V(x(\tau))\, d\tau]$, where $V(x)$ is a Borel measurable, real-valued function on R^n and $t > 0$. We interpret the integral in the following way: let

(3) For $\sigma = 0$ this is just the Wiener process.

(4) Quotation marks will be placed around terms which are not defined precisely in this paper. References will be given to more thorough treatments.

$$V_N(x) = \begin{cases} V(x), & V(x) \geqq -N, \\ -N, & V(x) < -N, \end{cases}$$

and define

$$-\int_0^t V(x(\tau))\,d\tau = \lim_{N\to\infty} \int_0^t -V_N(x(\tau))\,d\tau$$

unless $\int_0^t V_N(x(\tau))d\tau = \infty$ for some N, in which case let $-\int_0^t V(x(\tau))\,d\tau = -\infty$.

C. $F^3_{z,b_t}(x(\cdot)) \equiv \exp\left[-z\left(\int_0^t b(x(\tau))\right)\cdot dx + \frac{1}{2}\int_0^t \operatorname{div} b(x(\tau))\,d\tau\right],$

where $b(x)$ is a twice continuously differentiable vector field on R^n such that $\int e^{-\alpha x^2}b^2(x)\,dx < \infty$ for each $\alpha > 0$, z is a complex number and $t > 0$. (For $x, y \in R^n$ $x\cdot y = \sum_{j=1}^n x_j y_j$ and we denote $x\cdot x$ by x^2. Also, $\operatorname{div} b \equiv \sum_{j=1}^n \partial_j b_j$.) $\int_0^t b(x(\tau))\cdot dx$ denotes the Itô stochastic integral of $b(x)$ with respect to the n-dimensional Wiener process $x(\tau)$, $0 \leqq \tau < \infty$.

D. $F^4_{G_t}(x(\cdot)) \equiv I_{G_t}(x(\cdot))$, where G is an open subset of R^n, $\bar{G}$ is the closure of G, $G_t = \{x(\cdot) \in W \mid x(\tau) \in \bar{G},\ 0 \leqq \tau \leqq t\}$, I_{G_t} is the indicator (characteristic function) of G_t and $t > 0$.

$F^1_{\phi_t}$ is clearly $\mathfrak{N}$-measurable. See [7] and [8] for a discussion of $F^2_{V_t}$. (The case treated here is slightly more general but adds no new difficulties.) See [5] and [12] for a discussion of the stochastic integral. The $\mathfrak{N}$-measurability of F^3_{z,b_t} will then follow immediately. See [9] for a discussion of $F^4_{G_t}$.

If F is an $\mathfrak{N}$-measurable functional, then the integral of F with respect to P^0_x will be denoted by $E_x(F)$, if it exists. This, of course, is the Wiener integral. If $\hat{F}$ is an $\mathfrak{N}$-measurable functional, then the integral of $\hat{F}$ with respect to P^σ_x will be denoted by $E^\sigma_x(\hat{F})$, if it exists. $F^1_{\phi_t}$, $F^2_{V_t}$, $F^4_{G_t}$ and $\exp[-z\int_0^t \operatorname{div} b(x(\tau))\,d\tau]$ are, in fact, all $\mathfrak{N}$-measurable (see [7], [8], and [9]).

Let ϕ, V, b, G be as in A, B, C, D, respectively. We then define $(T_t(zb; V; G)\phi)(x)$ as follows:

$$(T_t(zb; V; G)\phi)(x) = E_x(F^1_{V_t}\cdot F^2_{V_t}\cdot F^3_{z,b_t}\cdot F^4_{G_t}), \tag{2.1}$$

when the integral exists. It will exist, for example, when V is bounded from below. When $G = R^n$ we will omit G and write $(T_t(zb; V)\phi)(x)$. This symbolism will be used for the remainder of the paper.

3. **Semigroup theory.** If $\{T_t \mid t \geqq 0\}$ is a strongly continuous semigroup of operators on a Banach space B and if Ω is its infinitesimal generator, then we will denote T_t by $\exp[\Omega_t]$. It is a fact that

$$R_\lambda(\Omega)\phi \equiv (\lambda I - \Omega)^{-1}\phi = \int_0^\infty e^{-\lambda t}T_t\phi\,dt,$$

for $\phi \in B$, $\mathrm{Re}(\lambda) > w_0 = \lim_{t\to\infty} (1/t) \log |T(t)|$. Moreover, the range of $R_\lambda(\Omega)$ is the domain of Ω and $(\lambda - \Omega)R_\lambda(\Omega) = I$. In §6 we will be interested in the resolvent of $-\Omega$, where Ω is the infinitesimal generator of a strongly continuous semigroup and, in particular, we will need the fact that

$$R_{-\lambda}(-\Omega) = -R_\lambda(\Omega) \tag{3.1}$$

for $\mathrm{Re}(\lambda) > w_0$. See [3, Chapter VIII], for a discussion of semigroup theory suitable for our purpose.

4. **Quadratic forms in Hilbert space.** Let H be a complex Hilbert space with inner product $(\cdot,\cdot)$. Let $\mathfrak{D}$ be a dense linear subset of H. A complex-valued function $J[\phi,\psi]$, defined for $\phi, \psi \in \mathfrak{D}$, is called a Hermitian bilinear form if $J[\phi,\psi] = J[\overline{\psi,\phi}]$ and $J[\cdot,\psi]$ is a linear functional on H for each $\psi \in H$. The complex conjuate of z is denoted by $\bar{z}$. $\mathfrak{D}$ is called the domain of J and will be denoted by $\mathfrak{D}(J)$. Associated with $J[\cdot,\cdot]$ is a quadratic form $J[\cdot]$ defined by $J[\phi] = J[\phi,\phi]$ for $\phi \in \mathfrak{D}(J)$. It is a fact that $J[\cdot]$ determines $J[\cdot,\cdot]$ and, hereafter, we will only consider quadratic forms. (We will drop "quadratic" and refer only to forms.) The notion of closeable and closed forms will be assumed to be known as well as the fact that semi-bounded self-adjoint operators on H and semi-bounded closed forms on H are in a natural one-to-one correspondence.

If J is a closeable form, then $\tilde{J}$ will denote its closure. If A is a semi-bounded symmetric operator on H with domain $\mathfrak{D}(A)$, then $J[\phi] = (A\phi,\phi)$, $\phi \in \mathfrak{D}(A)$, is a closeable form and we denote its closure by $J_{\tilde{A}}$, where $\tilde{A}$ is uniquely determined by $\tilde{J}$ (see the preceding paragraph). $\tilde{A}$ is a self-adjoint extension of A called the Friedrich's extension of A. See [15] for a thorough treatment of these topics.

If J_1, J_2 are forms on H, then we will write $J_1 \subset J_2$ if $\mathfrak{D}(J_1) \subset \mathfrak{D}(J_2)$ and $J_1[\phi] = J_2[\phi]$ for $\phi \in \mathfrak{D}(J_1)$. We write $J_1 \geqq J_2$ if $\mathfrak{D}(J_1) \subset \mathfrak{D}(J_2)$ and $J_1[\phi] \geqq J_2[\phi]$ for $\phi \in \mathfrak{D}(J_1)$. It is a fact that if $J_1 \leqq J_2$ and J_1 and J_2 are closeable then, $\tilde{J}_1 \leqq \tilde{J}_2$. If $\langle J_n \rangle$ is a sequence of forms, we write $J_n \uparrow$ if $J_n \leqq J_{n+1}$, $n = 1, 2, \cdots$ and $J_n \downarrow$ if $J_n \geqq J_{n+1}$, $n = 1, 2, \cdots$.

5. **Theorems and remarks.**

Notation. C_0^∞ will denote the infinitely differentiable complex-valued functions on R^n with compact support. If $V(x)$ is a Borel measurable function on R^n, let $\mathfrak{D}(V) = \{\phi \in L_2(R^n) \mid V\phi \in L_2(R^n)\}$. If ϕ, $\psi \in L_2(R^n)$, let $(\phi,\psi) = \int \phi\bar{\psi}\, dx$.

THEOREM 1. *Let $V(x)$ be as in the definition of $F^2_{V_t}$. Let $b(x)$ be as in the definition of F^3_{z,b_t}* (see §2). *Suppose that $D = C_0^\infty \cap \mathfrak{D}(V)$ is dense in $L_2(R^n)$, and that there exists $k > -\infty$ such that*

$$(A(0;V;D)\phi,\phi) \geqq k(\phi,\phi) \tag{5.1}$$

for all $\phi \in D$ and where $A(0;V;D)$ is defined by (1.1).

If $\tilde{A} = \tilde{A}(b; V; D)$ *denotes the Friedrich's extension of* $A(b; V; D)$, *then* $\exp[-\tilde{A}t]$ *can be represented as follows*:

(5.2) $$(\exp[-\tilde{A}t]\phi)(x) = (T_t(ib; V)\phi)(x) \text{ for } \phi \in L_2(R^n).$$

REMARK 5.1. When a self-adjoint operator is bounded above (such as $-\tilde{A}$) it follows directly from spectral theory that it is the infinitesimal generator of a strongly continuous self-adjoint semigroup.

REMARK 5.2. In studying a quantum mechanical system governed by a scalar potential $V(x)$ and a vector potential $b(x)$, there is some question as to which "self-adjoint extension" of $A(b; V; D)$ one takes as the "energy operator" for the system. (Of course, when $A(b; V; D)$ is essentially self-adjoint there is no problem (see [11]).) If we define the Feynman integral as the analytic extension in time t of the Wiener integral (5.2), then, from Theorem 1, we see that the Friedrich's extension of $A(b; V; D)$ is automatically singled out as the "energy operator." See [2] and [7] for the definition and discussion of this version of the Feynman integral.

REMARK 5.3. If V is a measure of "finite energy" on R^n, then it is still possible to give a meaning to $\int_0^t V(x(\tau))\, d\tau$ (see [6]) and thus the analysis of this paper should apply to such "scalar potentials" provided the "associated quadratic form" is still semi-bounded. However, we will not pursue this problem at this time.

In the following, G will be an open subset of R^n whose boundary has (Lebesgue) measure zero. Moreover, $L_2(G)$ will be considered as a closed subspace of $L_2(R^n)$ in the standard way.

THEOREM 2. *Let* $V(x)$ *and* $b(x)$ *be as in Theorem* 1. *Let*

$$\mathfrak{D}_G(V) = \{\phi \in L_2(G) \mid V\phi \in L_2(G)\}$$

and assume $D_G = \mathfrak{D}(\tilde{A}(b; 0; C_0^\infty)) \cap \mathfrak{D}_G(V)$ *is dense in* $L_2(G)$ *and that there exist* $k > -\infty$ *such that* $(\{\tilde{A}(b; 0; C_0^\infty) + V\}\phi, \phi) \geqq k(\phi, \phi)$ *for all* $\phi \in D_G$.

If $\tilde{A}_G$ *denotes the Friedrich's extension of* $\tilde{A}(b; 0; C_0^\infty) + V$ *considered as a symmetric semi-bounded operator on* $L_2(G)$ *with domain* D_G, *then* $\exp[-\tilde{A}_G t]$ *can be represented as follows*:

$$(\exp[-\tilde{A}_G t]\phi)(x) = (T_t(ib; V; G)\phi)(x) \text{ for } \phi \in L_2(G).$$

REMARK 5.4. As in Theorem 1, $\tilde{A}(b; 0; C_0^\infty)$ denotes the Friedrich's extension of $A(b; 0; C_0^\infty)$. In this special case the Friedrich's extension is merely the closure of $A(b; 0; C_0^\infty)$ (see [11]). Without loss of generality we can and do assume $V(x) = 0$ for $x \notin \bar{G}$.

REMARK 5.5. It should be remarked that the Friedrich's extension of a semi-bounded symmetric operator can be constructed in an explicit manner (see, e.g., [3, XII. 5] or [15]). This is what was meant when we stated in the introduction that, "$\tilde{A}$ will be given explicitly."

6. Proof of Theorems 1 and 2.

Proof of Theorem 1. We will first prove three lemmas which are special cases of the theorem.

LEMMA 1. *Let $V(x)$ be a bounded, Borel measurable real-valued function on R^n. Let $b(x) = (b_1(x), \cdots, b_n(x))$ be a twice continuously differentiable vector field on R^n which has compact support.*

Then $\widetilde{A}(b; V; C_0^\infty) = \widetilde{A}$, the Friedrich's extension of $A(b; V; C_0^\infty)$, is the L_2-closure of $A(b; B; C_0^\infty)$ and $\exp[-\widetilde{A}t]$ can be represented as follows:

$$(\exp[-\widetilde{A}t]\phi)(x) = (T_t(ib; V)\phi)(x) \text{ for } \phi \in L_2(R^n). \tag{6.1}$$

Proof. $A(b; V; C_0^\infty)$ is clearly semi-bounded (from below) and symmetric. Ikebe and Kato [11] have shown that it is essentially self-adjoint and, hence, the Friedrich's extension and L_2-closure coincide. Thus it remains to verify (6.1). This will be done in essentially two steps. We will first establish a formula similar to (6.1) for $T_t(\sigma b; V)$, where σ is real. Then, by analytic extension arguments, we will verify (6.1).

Let σ be a real number and $\phi \in L_2(R^n)$. It is known (see [4] and [17]) that

$$(T_t(\sigma b; V)\phi)(x) = E_x^\sigma\left\{ \exp\left[\frac{1}{2}\sigma^2 \int_0^t b^2(x(\tau))\, d\tau - \frac{1}{2}\sigma \int_0^t \operatorname{div} b(x(\tau))\, d\tau - \int_0^t V(x(\tau))\, d\tau \right]\phi(t) \right\}. \tag{6.2}$$

Moreover, it is known that the right-hand side of (6.2) defines a strongly continuous semigroup on $L_2(R^n)$ whose infinitesimal generator is

$$\Phi_\sigma = \Omega_\sigma + \frac{1}{2}\sigma^2 b^2(x) - \frac{1}{2}\sigma \operatorname{div} b - V(x), \qquad \mathfrak{D}(\Phi_\sigma) = \mathfrak{D}(\Omega_\sigma),$$

and where Ω_σ is the infinitesimal generator of the strongly continuous semigroup U_t^σ defined as follows:

$$(U_t^\sigma \phi)(x) = \int p^\sigma(t, x, y)\phi(x)\, dy$$

for $\phi \in L_2(R^n)$ (see [8]). The next step is to explicity determine Ω_σ (and, hence, $\mathfrak{D}(\Omega_\sigma)$).

If $\phi \in C_0^\infty$ it is known that $\phi(x, t) \equiv (U_t^\sigma \phi)(x)$ is twice continuously differentiable in $x = (x_1, \cdots, x_n)$, once continuously differentiable in t $(t > 0)$ and is a solution of

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\Delta\phi - \sigma(b \cdot \nabla)\phi, \quad \|\phi(x, t) - \phi(x)\|_2 \to 0 \quad \text{as } t \to 0, \tag{6.3}$$

where $\| \quad \|_2$ is the usual L_2-norm (see [3], §8).

On the other hand, if $\hat{\Delta}$ denotes the L_2-closure of Δ considered as an operator on C_0^∞ and $\hat{\partial}_j$, $j = 1, \cdots, n$, denotes the L_2-closure of ∂_j, $j = 1, \cdots, n$, considered as an operator on C_0^∞, then

$$\hat{\Omega}_\sigma = \frac{1}{2}\hat{\Delta} - \sigma(b \cdot \hat{\nabla}),$$

with $\mathfrak{D}(\hat{\Omega}_\sigma) = \mathfrak{D}(\hat{\Delta})$, is the infinitesimal generator of a strongly continuous semigroup on $L_2(R^n)$ and $\hat{\phi}(x,t) = (\exp[\hat{\Omega}_\sigma t]\phi)(x)$, $\phi \in C_0^\infty$, is a solution of

$$\frac{\partial \hat{\phi}}{\partial t} = \frac{1}{2}\hat{\Delta}\hat{\phi} - \sigma(b \cdot \hat{\nabla})\hat{\phi}, \quad \|\hat{\phi}(x,t) - \phi(x)\|_2 \to 0 \quad \text{as } t \to 0.$$

These results follow directly from the Phillip's perturbation formula (see [3, Theorem VIII. 1.19]) which is applicable here since $\mathfrak{D}(b \cdot \hat{\nabla}) \supseteq \mathfrak{D}(\hat{\Delta})$ and, for $\psi \in L_2(R^n)$,

$$\left\| \hat{\partial}_j \int p^0(t,x,y)\psi(y)\, dy \right\|_2 \leqq \frac{4\|\psi\|_2}{\sqrt{(2\pi t)}}, \qquad j = 1, \cdots, n.$$

Because of the special properties of the "unperturbed" semigroup U_t^0 which is generated by $p^0(t,x,y) = (2\pi t)^{-n/2}\exp[-(x-y)^2/2t]$, we have, in addition, from the Phillip's perturbation formula, that $\hat{\phi}(x,t)$ is twice continuously differentiable in x, once continuously differentiable in t and thus actually is a solution of (6.3). Hence from standard uniqueness theorems (see [3, Theorem XIV. 8.1]) we have $\phi(x,t) = \hat{\phi}(x,t)$, $t > 0$, and, since C_0^∞ is dense in $L_2(R^n)$, we have $U_t^\sigma = \exp[\hat{\Omega}_\sigma t]$ or, equivalently, that $\Omega_\sigma = \hat{\Omega}_\sigma$. From the remark just after (6.2), we have

$$(6.4) \qquad (T_t(\sigma b;\, V)\phi)(x) = (\exp[\Phi_\sigma t]\phi)(x),$$

where

$$\Phi_\sigma = \frac{1}{2}\hat{\Delta} - \sigma b \cdot \hat{\nabla} - \frac{1}{2}\sigma \operatorname{div} b + \frac{1}{2}\sigma^2 b - V,$$

$\mathfrak{D}(\Phi_\sigma) = \mathfrak{D}(\hat{\Delta})$, σ is real and $\phi \in L_2(R^n)$.

Again applying the Phillip's perturbation to the right-hand side of (6.4) (treating $-\sigma b \cdot \hat{\nabla} - \frac{1}{2}\sigma \operatorname{div} b + \frac{1}{2}\sigma^2 b^2 - V$ as the perturbation), we see that it can be extended to an entire (vector-valued) function in σ for fixed t and ϕ. Setting $\sigma = i$, we see that $\Phi_i = -\widetilde{A}(b;\, V;\, C_0^\infty)$. Thus we will be finished if we can show $T_t(\sigma b;\, V)\phi$ is also an entire (vector-valued) function in σ for fixed t and $\phi \in L_2\ R^n)$.

For σ a complex number we have

$$(6.5) \qquad \begin{aligned} E_x\Bigg\{ \Bigg| \exp\Bigg[-\Bigg(\sigma \int_0^t b(x(\tau)) \cdot dx + \frac{1}{2}\sigma \int_0^t \operatorname{div} b(x(\tau))\, d\tau \\ + \int_0^t V(x(\tau))\, d\tau \Bigg) \Bigg] \Bigg| \; |\phi(x(t))| \Bigg\} \\ \leqq \exp\left[\frac{1}{2}(\operatorname{Re}\sigma)^2 K_1 + K_2|\operatorname{Re}\sigma| + K_3)t \right] U_t^{\operatorname{Re}(\sigma)}|\phi|(x), \end{aligned}$$

where $K_1 = \sup_{x \in R^n} b^2(x)$, $K_2 = \sup_{x \in R^n} |\operatorname{div} b(x)|$, and $K_3 = \sup_{x \in R^n} |V(x)|$. Thus if we consider

$$\oint (T_t(\epsilon b;\, V)\phi, \psi)\, d\sigma,$$

for ϕ, $\psi \in L_2(R^n)$, we see from (6.5) that the (contour) integral commutes both with the inner product and the Wiener integral $(T_t(\sigma b;\, V)\phi)(x)$ and thus

$$\oint (T_t(\sigma b;\, V)\phi, \psi)\, d\sigma = 0, \tag{6.6}$$

since the integrand of the Wiener integral is an entire function in σ (with probability 1). Thus (6.6) implies $T_t(\sigma b;\, V)\phi$ is an entire function in σ for fixed t and $\phi \in L_2(R^n)$. Combining this with the preceding paragraph, we have

$$(T_t(ib;\, V)\phi)(x) = (\exp[\Phi_i t]\phi)(x) = (\exp[-\tilde{A}t]\phi)(x),$$

which was to be proved.

REMARK 6.1. Prokhorov's result (6.2) is a probabilistic result and, consequently, depends on the fact that σ is real. Thus we were forced to get to the case $\sigma = i$ rather indirectly. It would be nice to have a proof that avoids analytic extension arguments and treats $T_t(ib;\, V)$ directly. It is not hard to show directly that $T_t(ib;\, V)$ is a strongly continuous semigroup but to show it is self-adjoint and to determine the infinitesimal generator seem to be more difficult.

LEMMA 2. *Let $V(x)$ be as in Lemma 1 and $b(x)$ as in Theorem 1. Then $\tilde{A}(b;\, V;\, C_0^\infty) = \tilde{A}$, the Friedrich's extension of $A(b;\, V;\, C_0^\infty)$, is the L_2-closure of $A(b;\, V;\, C_0^\infty)$, and $\exp[-\tilde{A}t]$ can be represented as follows:*

$$(\exp[-\tilde{A}t]\phi)(x) = (T_t(ib;\, V)\phi)(x)$$

for $\phi \in L_2(R^n)$.

Proof. Since $A(b;\, V;\, C_0^\infty)$ is essentially self-adjoint (see [11]) the L_2-closure of $A(b;\, A;\, C_0^\infty)$ is also the Friedrich's extension $\tilde{A}(b;\, V;\, C_0^\infty)$. Thus it remains to verify the representation of $\exp[-\tilde{A}t]$.

The main idea of the proof is to approximate $b(x)$ by bounded vector fields $b^m(x)$, and then pass to the limit using Lemma 1, bounded convergence theorems, and the fact that the "limit" of self-adjoint semigroups is a self-adjoint semigroup.

Let $\langle b^m(x)\rangle$, $m = 1, 2, \cdots$, be a sequence of bounded twice continuously differentiable vector fields on R^n such that $b^m(x) = b(x)$ for $x^2 \leqq m^2$, $(b^m(x))^2 \leqq b^2(x)$ and $b^m(x)$ vanishes outside a compact set[5]. Thus, $\lim_{m \to \infty} b_j^m(x)$

[5] The construction of such $b^m(x)$ would be tedious but straightforward and will not be given here.

$= b_j(x)$, $j = 1, \cdots, n$, and all $x \in R^n$ and, hence,

$$\lim_{m\to\infty} \int_0^t b^m(x(\tau)) \cdot dx = \int_0^t b(x(\tau)) \cdot dx, \tag{6.7}$$

in P_x^0 measure, for each $x \in R^n$ (see [5, §3, Remark 2]). From (6.7) it clearly follows that

$$\lim_{m\to\infty} F^3_{i,b_t^m} = F^3_{i,b_t}$$

in P_x^0 measure, for each $x \in R^n$. This, combined with the fact that

$$|F^1_{\phi_t}(x(\cdot)) \cdot F^2_{V_t}(x(\cdot)) \cdot F^3_{i,b_t}(x(\cdot))| \leqq \exp[Kt]|\phi(x(t))|,$$

where $K = \sup_{x\in R^n}|V(x)|$ and $\phi \in L_2(R^n)$, implies that

$$\lim_{m\to\infty} (T_t(ib^m; V)\phi)(x) = (T_t(ib; V)\phi)(x)$$

for each $x \in R^n$. This is just an application of the dominated convergence theorem for abstract integrals. Not also that

$$|(T_t(ib^m; V)\phi)(x)|^2 \leqq 2^{Kt}|E_x\{|\phi(x(t))|\}|^2$$

and, thus,

$$\lim_{m\to\infty} \| T_t(ib^m; V)\phi - T_t(ib; V)\phi\|_2 = 0 \tag{6.8}$$

and

$$\| T_t(ib^m; V)\phi\|_2 \leqq 2e^{Kt}\|\phi\|_2 \tag{6.9}$$

for $t \geqq 0$ and $\phi \in L_2(R^n)$. It follows from general operator theory that $T_t(ib; V)$ is a self-adjoint semigroup which is bounded on $[0,t]$ by $2e^{Kt}$. In fact, from Lemma 22.3.1 in [10], it follows that $T_t(ib; V)$ is strongly continuous on $(0, \infty)$.

It remains to be shown that $T_t(ib; V)$ is strongly continuous (on $[0, \infty)$) and has $-\tilde{A}$ as infinitesimal generator. Let

$$R_\lambda^m\phi = \int_0^\infty e^{-\lambda t} T_t(ib^m; V)\phi \, dt; \qquad R_\lambda\phi = \int_0^\infty e^{-\lambda t} T_t(ib; V)\phi \, dt$$

for $\mathrm{Re}\,\lambda > K$ and $\phi \in L_2(R^n)$. The integrals are interpreted as Bochner integrals. From (6.8) and (6.9) it follows that

$$\lim_{m\to\infty} \| R_\lambda^m\phi - R_\lambda\phi\|_2 = 0$$

for $\phi \in L_2(R^n)$. This implies that the infinitesimal generator Ω of $T_t(ib; V)$ is an extension of $-A(b; V; C_0^\infty)$. The proof of this fact will be omitted since it is essentially the same as the proof of an analogous result given in [8, p. 1590]. Since the range of R_λ is the domain of Ω it follows that the range of R_λ is dense in $L_2(R^n)$ and, thus, by Theorem 22.3.2 in [10], $T_t(ib; V)$

is strongly continuous on $[0,\infty)$ and Ω is a self-adjoint extension of $-A(b;V;C_0^\infty)$. Since $A(b;V;C_0^\infty)$ is essentially self-adjoint, $\Omega = -\widetilde{A}$, which completes the proof of the lemma.

Lemma 3. *Let V, b and D be as in Theorem 1 and, for $N = 1,2,\cdots$, let*

$$V_N(x) = \begin{cases} V(x), & \text{if } V(x) > -N, \\ -N, & \text{if } V(x) \leqq -N. \end{cases}$$

If $\widetilde{A}(b;V_N;D) = \widetilde{A}_N$ denotes the Friedrich's extension of $A(b;V_N;D)$, then $\exp[-\widetilde{A}_N t]$ can be represented as follows:

$$(\exp[-\widetilde{A}_N t]\phi)(x) = (T_t(ib;V_N)\phi)(x)$$

for $\phi \in L_2(R^n)$.

Proof. For $m = 1,2,\cdots$, let

$$V_N^m(x) = \begin{cases} V_N(x), & \text{if } V_N(x) < m, \\ m, & \text{if } V_N(x) \geqq m. \end{cases}$$

By using essentially the same arguments that were used in the proof of Lemma 2, we see that $T_t(ib;V_N)$ is a strongly continuous self-adjoint semigroup whose infinitesimal generator Ω_N is a self-adjoint extension of $-A(b;V_N;D)$ and

$$\lim_{m\to\infty} \| R_\lambda(-\widetilde{A}(b;V_N^m;C_0^\infty))\phi - R_\lambda(\Omega_N)\phi\|_2 = 0 \tag{6.10}$$

for $\phi \in L_2(R^n)$ and $\operatorname{Re}\lambda > N$.

It remains to be shown that $\Omega = -\widetilde{A}_N$. To accomplish this we will use quadratic forms in Hilbert space. For $\phi \in C_0^\infty$ define the quadratic forms $J_m[\phi] = (A(b;V_N^m;C_0^\infty)\phi,\phi)$, $m = 1,2,\cdots$, and for $\phi \in D$, define the quadratic form $J[\phi] = (A(b;V;D)\phi,\phi)$. Clearly $J_m \leqq J$ and $J_m \uparrow$ and thus $\widetilde{J}_m = J_{\widetilde{A}(b;V_N^m;C_0^\infty)} \leqq \widetilde{J} = J_{\widetilde{A}_N}$ and $\widetilde{J}_m \uparrow$. From (3.1) and (6.10) we see that

$$\lim_{m\to\infty} \| R_\lambda(\widetilde{A}(b;V_N^m;C_0^\infty))\phi - R_\lambda(-\Omega)\phi\|_2 = 0$$

for $\phi \in L_2(R^n)$ and $\operatorname{Re}\lambda < -N$. Thus, from Theorem 10.1, part i, in [15] we conclude $\lim_{m\to\infty} \widetilde{J}_m = J_{-\Omega} \leqq J_{\widetilde{A}_N}$. On the other hand, by a direct calculation we observe that $\sup \widetilde{J}_m \supset J_{\widetilde{A}_N}$, where $\mathfrak{D}(\sup \widetilde{J}_m) = \{\phi \in C_0^\infty \mid \lim_{m\to\infty} \widetilde{J}_m[\phi] \text{ exists}\}$ and $\sup \widetilde{J}_m[\phi] = \lim_{m\to\infty} \widetilde{J}_m[\phi]$. Thus, from Theorem 10.1, part ii, in [15] we have that $J_{\widetilde{A}_N} \leqq J_{-\Omega}$ and, thus, $-\Omega = \widetilde{A}_N$ or, equivalently, $\Omega = -\widetilde{A}_N$, which completes the proof of the lemma.

We will now conclude the proof of Theorem 1. First note that if $V_N(x)$ is as in Lemma 3, then

$$\| T_t(0;V_N)\|_2 \leqq e^{|k|t}\|\phi\|_2 \tag{6.11}$$

for $\phi \in L_2(R^n)$. This follows from the fact that $-A(0;V_N;D)$ and, hence, $-\widetilde{A}(0;V_N;D)$ are bounded above by $|k|$ (see 5.1) and thus

$$T_t(0;\, V_N)\phi = \int_{-\infty}^{|k|} e^{\lambda t}\, dE_\lambda^N \phi, \tag{6.12}$$

where $\{E_\lambda^N\}$ is the spectral resolution of the identity associated with $-\tilde{A}(0;\, V_N;\, D)$. The equation (6.12) clearly implies (6.11).

In addition, note that, for $\phi \in L_2(R^n)$,

$$|(T_t(ib;\, V_N)\phi)(x)| \leqq \{(T_t(0;\, V)|\phi|)(x)\} \in L_2(R^n); \tag{6.13}$$

$$\lim_{N\to\infty} (T_t(ib;\, V_N)\phi)(x) = (T_t(ib;\, V)\phi)(x) \tag{6.14}$$

for almost all x (in the sense of Lebesgue measure) and thus

$$\| T_t(ib;\, V_N)\phi \|_2 \leqq e^{|k|t} \|\phi\|_2; \tag{6.15}$$

$$\lim_{N\to\infty} \| T_t(ib;\, V_N)\phi - T_t(ib;\, V)\phi \|_2 = 0. \tag{6.16}$$

Equation (6.13) follows (6.11) and the fact that

$$|(T_t(ib;\, V_N)\phi)(x)| \leqq (T_t(0;\, V_N)|\phi|)(x) \uparrow (T_t(0;\, V)|\phi|)(x).$$

Equation (6.13) in turn implies $0 \leqq (T_t(0;\, V)|\phi|)(x) < \infty$ for almost all x which, combined with the fact that

$$|F^1_{\phi_t}(x(\cdot)) \cdot F^2_{V_{N_t}}(x(\cdot)) \cdot F^3_{i,b_t}(x(\cdot))| \leqq \exp\left[-\int_0^t V(x(\tau))\, d\tau \right] |\phi(x(t))|,$$

implies (6.14).

Just as in the proof of Lemma 2, (6.15) and (6.16) imply that $T_t(ib;\, V)$ is a strongly continuous self-adjoint semigroup whose infinitesimal generator Ω is a self-adjoint extension of $-A(b;\, V;\, D)$. Moreover,

$$\lim_{N\to\infty} \| R_\lambda(-\tilde{A}(b;\, V_N;\, D)\phi - R_\lambda(\Omega))\phi \|_2 = 0 \tag{6.17}$$

for $\phi \in L_2(R^n)$ and $\operatorname{Re}\lambda < |k|$.

It remains to be shown that $\Omega = -\tilde{A}(b;\, V;\, D)$. For $\phi \in D$, define the quadratic forms $J_N[\phi] = (A(b;\, V_N;\, D)\phi, \phi)$ for $N = 1, 2, \cdots$ and $J[\phi] = (A(b;\, V;\, D)\phi, \phi)$. Clearly $J_N \geqq J$ for all N and $J_N \downarrow$. From (3.1), (6.17) and Theorem 10.2 in [15] we have that $J_{-\Omega} \geqq J_{\tilde{A}(b;\, V;\, D)}$. On the other hand, both $-\Omega$ and $\tilde{A}(b;\, V;\, D)$ extend $A(b;\, V;\, D)$ and, since $\tilde{A}(b;\, V;\, D)$ is the Friedrich's extension of $A(b;\, V;\, D)$, we have, by Theorem 8.3 in [15], that $J_{-\Omega} \leqq J_{\tilde{A}(b;\, V;\, D)}$ and thus $\Omega = -\tilde{A}(b;\, V;\, D)$, which completes the proof of Theorem 1.

Proof of Theorem 2. We will only give the details of the proof for the case where $\sup_{x\in\bar{G}} |V(x)| = K < \infty$. The proof of the general case will then follow closely the proof of Theorem 1 after Lemmas 1 and 2 had been established.

Thus, for the remainder of the proof, we will assume $|V(x)|$ is bounded in $\bar{G}$ by K and let

$$V_n(x) = \begin{cases} V(x), & x \in \bar{G}, \\ n, & x \notin \bar{G}. \end{cases}$$

Reasoning as in [9, Theorem 4.1], we see that

$$\lim_{n\to\infty} \| T_t(ib;\, V_n)\phi - T_t(ib;\, V;\, G)\phi \|_2 = 0 \tag{6.18}$$

and

$$T_t(ib;\, V;\, G) \cdot \chi_G = \chi_G \cdot T_t(ib;\, V;\, G),$$

where χ_G is the natural projection of $L_2(R^n)$ on $L_2(G)$ and $\phi \in L_2(R^n)$. Thus from (6.18), Lemma 2 and the fact that

$$\| T_t(ib;\, V_n)\phi \|_2 \leqq e^{Kt} \cdot \| \phi \|_2, \tag{6.19}$$

we conclude that $T_t(ib;\, V;\, G)$ is a self-adjoint semigroup of operators on $L_2(g)$ which is bounded on bounded sets of t. It will be strongly continuous on $[0, \infty)$ if the range of

$$R_\lambda = \int_0^\infty e^{-\lambda t} T_t(ib;\, V;\, G)\, d\lambda,$$

for $\operatorname{Re}\lambda > K$, is dense in $L_2(G)$ (by Theorem 22.3 in [10]). It is clear that $D_G \subseteq \mathfrak{D}(\tilde{A}(b;\, V_n;\, C_0^\infty))$ and thus, for $\phi \in D_G$, there exists $\psi_n \in L_2(R^n)$ such that

$$\phi = R_\lambda(-\tilde{A}(b;\, V_n;\, C_0^\infty))\psi_n$$

for $\operatorname{Re}\lambda > K$ and thus

$$\begin{aligned} \psi_n &= \{\lambda - (-\tilde{A}(b;\, V_n;\, C_0^\infty))\}\phi \\ &= (\lambda + \tilde{A}(b; 0;\, C_0^\infty) + V)\phi \equiv \psi \in L_2(G). \end{aligned} \tag{6.20}$$

But from (6.18) and (6.19) we have

$$\lim_{n\to\infty} \| R_\lambda(-\tilde{A}(b;\, V_n;\, C_0^\infty))\psi - R_\lambda\psi \|_2 = 0$$

and hence $R_\lambda\psi = \phi$. Since D_G is dense in $L_2(G)$ and we have just shown the range of R_λ contains D_G, we have that $T_t(ib;\, V;\, G)$ is a strongly continuous self-adjoint semigroup of operators on $L_2(G)$.

It remains to be shown that $\Omega = -\tilde{A}_G$, where Ω is the (self-adjoint) infinitesimal generator of $T_t(ib;\, V;\, G)$. Since $T_t(ib;\, V;\, G)$ is strongly continuous, $R_\lambda = R_\lambda(\Omega)$ which, combined with (6.20), imply that

$$(\lambda - \Omega)\phi = \psi = (\lambda + \tilde{A}(b; 0;\, C_0^\infty) + V)\phi$$

for $\phi \in D_G$ and thus $\Omega \supseteq -\tilde{A}_G$.

On the other hand, let $\tilde{J}$ be the closed form associated with $\tilde{A}(b;\, V;\, C_0^\infty)$

$+ V$ ($\widetilde{A}(b;0) + V$ is already self-adjoint since V is bounded) and let $\mathfrak{D}(J_G) = \{\phi \in \mathfrak{D}(\widetilde{J}) \mid \chi_G\phi = \phi\}$ and define $J_G[\phi] \equiv \widetilde{J}[\phi]$, for $\phi \in \mathfrak{D}(J_G)$. It is clear that J_G is a closed form in $L_2(G)$ and, in fact, can be represented as follows:

$$J_G[\phi] = (\{\widetilde{A}(b;0;C_0^\infty) + V - K\}^{1/2}\phi, \{\widetilde{A}(b;0;C_0^\infty) + V - K\}^{1/2}\phi) + K(\phi,\phi).$$

This implies (see [15, Theorem 4.2]) that $\{\widetilde{A}(b;0;C_0^\infty) + V - K\}^{1/2}$, with domain $\mathfrak{D}(J_G)$, is a self-adjoint operator on $L_2(G)$ and thus $\widetilde{A}(b;0;C_0^\infty) + V$ with domain D_G, i.e., $\widetilde{A}_G$, is a self-adjoint operator on $L_2(G)$. Since self-adjoint operators are maximal symmetric, we have $\Omega = -\widetilde{A}_G$ and Theorem 2 is proved.

Bibliography

1. D. G. Babbitt, *The Wiener integral and perturbation theory of the Schrödinger operator*, Bull. Amer. Math. Soc. **70** (1964), 254-259.

2. ———, *A summation procedure for certain Feynman integrals*, J. Mathematical Phys. **4** (1963), 36-41.

3. N. Dunford and J. Schwartz, *Linear operator*, Parts I, II, Interscience, New York, 1958; 1963.

4. E. B. Dynkin, *Transformations of Markov processes connected with additive functionals*, Proc. 4th Berkeley Sympos. Math. Statist. Prob., Univ. California Press, Berkeley, Calif., 1961, pp. 117-142.

5. ———, *Additive functionals of a Wiener process determined by stochastic integrals*, Theor. Probability Appl. **5** (1960), 402-411.

6. ———, *Markov processes and problems in analysis*, Amer. Math. Soc. Transl. (2) **31** (1963), 1-24.

7. J. Feldman, *On the Schrödinger and heat equations for non-negative potentials*, Trans. Amer. Math. Soc. **108** (1963), 251-264.

8. R. K. Getoor, *Additive functionals of a Markov process*, Pacific J. Math. **7** (1957), 1577-1591.

9. ———, *Markov operators and their associated semi-groups*, Pacific J. Math. **9** (1959), 449-472.

10. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, rev. ed., Amer. Math. Soc. Colloq. Publ. Vol. 31, Amer. Math. Soc., Providence, R. I., 1957.

11. T. Ikebe and T. Kato, *Uniqueness of the self-adjoint extension of singular elliptic differential operators*, Arch. Rational Mech. Anal. **9** (1962), 77-92.

12. K. Itô, *On stochastic differential equations*, Mem. Amer. Math. Soc. No. **4** (1951), 51 pp.

13. S. Itô, *Fundamental solutions of parabolic differential equations and boundary value problems*, Japan. J. Math. **27** (1957), 55-102.

14. M. Kac, *On some connections between probability theory and differential and integral equations*, Proc. 2nd Berkeley Sympos. Math. Statist. Prob., Univ. California Press, Berkeley, Calif., 1951.

15. T. Kato, *Quadratic forms in Hilbert space and asymptotic perturbation series*, Tech. Rep. No. 7, Department of Math., Univ. of California, Berkeley, Calif., 1955.

16. L. Landau and E. Lipshitz, *Quantum mechanics, non-relativistic theory*, Addison-Wesley, Reading, Mass., 1958.

17. Ju. Prokhorov, *Convergence of random processes and limit theorems in probability theory*, Appendix 2, Theor. Probability Appl. **1** (1956), 207-214.

18. D. Ray, *On spectra of second order operators*, Trans. Amer. Math. Soc. **77** (1954), 299-321.

19. K. Yosida, *On the fundamental solution of the parabolic equations in a Riemannian space*, Osaka Math. J. **5** (1953), 65-74.

University of California,
Los Angeles, California